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ON THE COMMUTATIVITY OF PRIME RINGS USING SOME DIFFERENTIAL IDENTITIES

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Abstract. Many research have been conducted to investigate the commutativity of prime rings utilizing some functional identities assertions, which shows that functional identities has a significant impact on the ring structures. Using certain differential identities, we extend the literature to demonstrate that prime rings that admit some generalized semi-derivations are integral domains.

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1. Introduction

Rings considered in this paper are associative and not necessarily unitary. We shall denote by $Z(R)$ the center of a ring R. An ideal P of R is a prime ideal if $xRy \subseteq P$ yields $x \in P$ or $y \in P$. In particular, if the zero ideal of R is prime, then R is said to be a *prime ring*. For any $x, y \in R$, we will write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. An additive mapping $d : R \to R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x,\ y \in R$. An additive mapping $F : R \to R$ is a *generalized derivation* associated to a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x,\ y \in R$. When it comes to inferring a derivation map associated with a given ring function, an additive mapping $f : R \to R$ is a *semi-derivation* associated with an epimorphism g, if $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ with $d(g(x)) = g(d(x))$ for all $x,\ y \in R$.

Combining the two previous definitions, one can be able to present the next derivation map. An additive mapping F on a ring R associated with a semi-derivation f and an epimorphism g, for which $F(g(x)) = g(F(x))$ and $F(xy) = F(x)y + g(x)f(y) = F(x)g(y) + xf(y)$, for all $x,\ y \in R$, is called a *generalized semi-derivation*.

Several innovations in literature reveal on how the behavior of some additive mappings on a prime ring effects on the structure of such rings. Many of the achieved findings expand the earlier ones to show the impact of the considered additive mappings on the entire ring, see [3, 4, 5, 6].

The second theorem of Posner stated that, a prime ring is commutative if it admits a nonzero centralizing derivation [7]. Considering such theorem from a

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distance, it is unclear what motivated Posner to prove it and for what argument he has been willing to consider that the theorem is valid. Certainly, Posner's theorem achieved a huge impact, and it has helped to launch a number of concepts. The most broad and significant of these is the concept of functional identity.

There is certainly a lot of interest in investigating the commutativity of rings, particularly the class of prime (semi-prime) rings. The current article will go over some assumptions on a selective additive mapping on prime rings which provides the commutativity of such ring. The selective additive mapping in this article is the generalized semi-derivation.

2. Prime rings admitting generalized semi-derivation

In the current section, $(F, \tilde{f}, f)$ for a given ring R denotes a generalized semi-derivation $F : R \to R$ associated with a semi-derivation f and an epimorphism $\tilde{f}$.

In [1], Ashraf et al. showed that a prime ring R is commutative corresponding to a non-zero ideal I of R, if R admits a generalized derivation F associated with a non-zero derivation d, such that $F(xy) + xy \in Z(R)$, for all $x,\ y \in I$. We will demonstrate the commutativity of prime rings under other derivation identities using the next theorem.

Theorem 2.1. *Let R be a prime ring and $(F, \tilde{f}, f)$ be a nonzero generalized semi-derivation of a prime ring R, such that $F(x) \circ y \in Z(R)$ for all $x,\ y \in R$. Then R is an integral domain.*

Proof. By assumption

$$[F(x) \circ y, r] = 0, \text{ for all } r, x, y \in R. \tag{1}$$

Changing y by ry in (1), we get

$$r[F(x) \circ y, r] + \big[[F(x), r], r\big]y + [F(x), r][y, r] = 0, \text{ for all } r, x, y \in R. \tag{2}$$

In view of (1), equation (2) reduces to

$$\big[[F(x), r], r\big]y + [F(x), r][y, r] = 0, \text{ for all } r, x, y \in R. \tag{3}$$

Replacing y by $yF(x)$ in (3), one can verify that

$$[F(x), r]R[F(x), r] = 0, \text{ for all } r, x \in R.$$

Invoking the primeness of R, we obtain

$$[F(x), r] = 0, \text{ for all } r, x \in R. \tag{4}$$

On the other hand, writing xy for x in (4), it follows that

$$F(x)[y, r] + \tilde{f}(x)[f(y), r] + [\tilde{f}(x), r]f(y) = 0, \text{ for all } r, x, y \in R. \tag{5}$$

Putting $F(x)$ instead of x in (5), we find that

$$F^2(x)[y, r] + F(\tilde{f}(x))[f(y), r] = 0, \text{ for all } r, x, y \in R. \tag{6}$$

Taking $y = r$ in (6), we may write

$$F(\tilde{f}(x))[f(r), r] = 0, \text{ for all } r, x \in R. \tag{7}$$

Let $f(y)$ instead of r and $\tilde{f}(x)$ instead of x in (5) and using equation (7), it follows that

$$[\tilde{f}^2(x), f(r)]f(r) = 0, \text{ for all } r, x \in R. \tag{8}$$

Since $\tilde{f}$ is an epimorphism, we obviously get

$$[x, f(r)]f(r) = 0, \text{ for all } r, x \in R, \tag{9}$$

which proves that

$$[x, f(r)]R[x, f(r)] = 0, \text{ for all } r, x \in R. \tag{10}$$

We conclude that $f(R) \subseteq Z(R)$, then relation (5) reduces to

$$F(x)[y, r] + [\tilde{f}(x), r]f(y) = 0 \text{ for all } r, x, y \in R.$$

In this case, taking $y = r$ in the latter equation, we acquire

$$[x, r]f(r) = 0, \text{ for all } r, x \in R. \tag{11}$$

Substituting xt for x in (11), we result

$$[x, r]Rf(r) = 0, \text{ for all } r, x \in R. \tag{12}$$

Therefore, R is commutative or $f = 0$ which implies from equation (5) that $F = 0$, a contradiction. □

Corollary 2.2. *Let R be a prime ring and F be a nonzero generalized derivation of R associated with a derivation f, such that $F(x) \circ y \in Z(R)$ for all $x, y \in R$. Then, R is an integral domain.*

Theorem 2.3. *Let R be a prime ring and $(F, \tilde{f}, f)$ be a nonzero generalized semi-derivation on R. The following assertions are equivalent:*

1. *$[F(x), y] \in Z(R)$, for all $x, y \in R$.*
2. *R is an integral domain.*

Proof. For the non trivial implication, we have that

$$\big[[F(x), y], r\big] = 0, \text{ for all } r, x, y \in R. \tag{13}$$

Changing y by ry in (13), we get

$$r\big[[F(x), y], r\big] + \big[[F(x), r], r\big]y + [F(x), r][y, r] = 0, \text{ for all } r, x, y \in R. \tag{14}$$

In view of (13), equation (14) reduces to

$$[F(x), r][y, r] = 0, \text{ for all } r, x, y \in R. \tag{15}$$

Replacing y by $yF(x)$ in (15), one can verify that

$$[F(x), r]R[F(x), r] = 0, \text{ for all } r, x \in R.$$

Invoking the primeness of R, we obtain

$$[F(x), r] = 0, \text{ for all } r, x \in R. \tag{16}$$

On the other hand, writing xy for x in (16), we get

$$F(x)[y,r] + \tilde{f}(x)[f(y),r] + [\tilde{f}(x),r]f(y) = 0, \text{ for all } r,x,y \in R. \tag{17}$$

Putting $F(x)$ instead of x in (17), we find that

$$F^2(x)[y,r] + F(\tilde{f}(x))[f(y),r] = 0, \text{ for all } r,x,y \in R. \tag{18}$$

Taking $y = r$ in (18), we may write

$$F(\tilde{f}(x))[f(r),r] = 0, \text{ for all } r,x \in R. \tag{19}$$

Let $f(y)$ instead of r and $\tilde{f}(x)$ instead of x in (17) and using equation (19), it follows that

$$[\tilde{f}^2(x), f(r)]f(r) = 0, \text{ for all } r,x \in R. \tag{20}$$

Since $\tilde{f}$ is an epimorphism, we obviously get

$$[x, f(r)]f(r) = 0, \text{ for all } r,x \in R, \tag{21}$$

which proves that

$$[x, f(r)]R[x, f(r)] = 0, \text{ for all } r,x \in R. \tag{22}$$

We conclude that $f(R) \subseteq Z(R)$, then relation (17) reduces to

$$F(x)[y,r] + [\tilde{f}(x),r]f(y) = 0, \text{ for all } r,x,y \in R.$$

In this case, taking $y = r$ in the latter equation, we acquire

$$[x,r]f(r) = 0, \text{ for all } r,x \in R. \tag{23}$$

Substituting xt for x in (23), we result

$$[x,r]Rf(r) = 0, \text{ for all } r,x \in R. \tag{24}$$

Therefore, R is commutative or $f = 0$ which implies from equation (17) that $F = 0$, a contradiction. □

Corollary 2.4. *Let R be a prime ring and F be a nonzero generalized derivation of a prime ring R associated with a derivation f, such that $[F(x), y] \in Z(R)$ for all $x, y \in R$. Then, R is an integral domain.*

Theorem 2.5. *Let R be a prime ring, $(F, \tilde{f}, f)$ and $(G, \tilde{g}, g)$ two generalized semi-derivations of R. If $F(x)G(y) \in Z(R)$ for all $x, y \in R$, then one of the following claims is valid:*

(1) *$F = 0$ or $G = 0$.*
(2) *There exist $\lambda, \lambda' \in C$ and two additive mappings $\mu, \mu' : R \to C$ such that $F(x) = \lambda x + \mu(x)$ and $G(x) = \lambda' x + \mu'(x)$.*
(3) *R is an integral domain.*

Proof. By hypothesis, we have

$$F(x)G(y) \in Z(R), \text{ for all } x,y \in R. \tag{25}$$

Replacing y by yr in (25), we get

$$F(x)G(y)\tilde{g}(r) + F(x)yg(r) \in Z(R), \text{ for all } r,x,y \in R. \tag{26}$$

Commuting the above relation with $\tilde{g}(r)$, we obtain

$$[F(x)yg(r), \tilde{g}(r)] = 0, \text{ for all } r, x, y \in R. \tag{27}$$

Substituting xs for x in (27), we find that

$$[F(x)\tilde{f}(s)yg(r), \tilde{g}(r)] + [xf(s)yg(r), \tilde{g}(r)] = 0, \text{ for all } r, s, x, y \in R. \tag{28}$$

Putting $\tilde{f}(s)y$ for y in (27) and subtracting from (28), we arrive at

$$[xf(s)yg(r), \tilde{g}(r)] = 0, \text{ for all } r, s, x, y \in R. \tag{29}$$

Writing tx for x in (29) and using it, one can see that

$$[t, \tilde{g}(r)]xf(s)yg(r) = 0, \text{ for all } r, s, t, x, y \in R. \tag{30}$$

Then $R = R_1 \cup R_2$ with $R_1 = \{r \in R \,|\, [t, \tilde{g}(r)] = 0\}$ and $R_2 = \{r \in R \,|\, f(s)yg(r) = 0\}$. Using Brauer's trick, we have $R = R_1$ or $R = R_2$.

If $R = R_1$, i. e. $\tilde{g}(R) \subset Z(R)$, then equation (26) becomes

$$[F(x)yg(r), s] = 0, \text{ for all } r, s, x, y \in R. \tag{31}$$

Writing $yg(r)$ for y in (31), one is able to verify that

$$F(x)yg(r)[g(r), s] = 0, \text{ for all } r, s, x, y \in R. \tag{32}$$

Substituting ts for s in (32), it follows that

$$F(x)yg(r)t[g(r), s] = 0, \text{ for all } r, s, t, x, y \in R. \tag{33}$$

Simple computations lead to

$$F(x)y[g(r), s]t[g(r), s] = 0, \text{ for all } r, s, t, x, y \in R. \tag{34}$$

Applying the primeness of R, we find that $F(R) = 0$ or $g(R) \subseteq Z(R)$. However, the second case together with equation (31) leads to the first case or $g(R) = 0$ or R is commutative.

Now if $R = R_2$, then we get $f(R) = 0$ or $g(R) = 0$. Suppose that $f = 0$, then putting $xG(y)$ for x in our assumption, we get

$$F(x)G(y)[G(y), s] = 0, \text{ for all } s, x, y \in R,$$

implies that

$$F(x)R[G(y), s]R[G(y), s] = 0, \text{ for all } s, x, y \in R.$$

Therefore, $F(R) = 0$ or $G(R) \subset Z(R)$. In the second case, using the hypothesis we get $F(R) \subset Z(R)$ or $G(R) = 0$. Now if $[F(x), x] = 0$ and $[G(x), x] = 0$ for all $x \in R$, then by ([2], Theorem 3.2) we obtain the required results. □

Corollary 2.6. *Let $(F, \tilde{f}, f)$ be a nonzero generalized semi-derivation of a prime ring R. If $F(x)\ F(y) \in Z(R)$, for all $x,\ y \in R$. Then one of the following claims is valid:*

(1) *$F(x) = \lambda x + \mu(x)$, for $\lambda \in C$ and $\mu : R \to C$.*
(2) *R is an integral domain.*

Theorem 2.7. *Let R be a prime ring and $(F, \tilde{f}, f)$ be a generalized semi-derivation on R associated to a nonzero derivation f, such that $f(Z(R)) \neq 0$. Then, the following assertions are equivalent:*

(1) $[F(x), f(y)] \in Z(R)$, *for all* $x, y \in R$.
(2) *R is an integral domain.*

Proof. For the non trivial implication, suppose that $Z(R) = (0)$, then the main equation reduces to

$$[F(x), f(y)] = 0, \text{ for all } x, y \in R. \tag{35}$$

Taking $yf(r)$ instead of y in (35), with $r \in R$, we get

$$f(y)[F(x), \tilde{f}(f(r))] + [F(x), yf^2(r)] = 0, \text{ for all } r, x, y \in R. \tag{36}$$

Using (35), equation (36) becomes

$$[F(x), yf^2(r)] = 0, \text{ for all } r, x, y \in R. \tag{37}$$

Accordingly

$$[F(x), y]Rf^2(r) = 0, \text{ for all } r, x, y \in R.$$

Invoking the primeness of R, we get either $[F(R), R] = (0)$ or $f^2(R) = (0)$. In light of Theorem 2.3, the first case implies that R is an integral domain.

Now, if

$$f^2(r) = 0, \text{ for all } r \in R. \tag{38}$$

Replacing r by rs, with $s \in R$, we get

$$f\big(f(r)\tilde{f}(s) + rf(s)\big) = 0, \text{ for all } r, s \in R.$$

That is

$$f^2(r)\tilde{f}^2(s) + f(r)f(\tilde{f}(s)) + f(r)f(s) + \tilde{f}(r)f^2(s) = 0, \text{ for all } r, s \in R.$$

In light of equation (38), we get

$$f(r)f(s + \tilde{f}(s)) = 0, \text{ for all } r, s \in R. \tag{39}$$

Using the fact that $\mathrm{id} + \tilde{f}$ is surjective, we obtain

$$f(r)f(x) = 0 \quad \text{for all } r, x \in R.$$

Replacing r by ry, with $y \in R$, we get

$$f(r)yf(x) + \tilde{f}(r)f(y)f(x) = 0 \quad \text{for all } r, x \in R.$$

Thus

$$f(r)Rf(x) = 0 \quad \text{for all } r, x \in R,$$

then $f = 0$, a contradiction.

Now if $Z(R) \neq (0)$, we have that $f(Z(R)) \neq 0$ along with

$$[F(x), f(y)] \in Z(R), \text{ for all } x, y \in R. \tag{40}$$

Replacing y by yz, with $z \in Z(R)\backslash\{0\}$ and $f(z) \neq 0$, we get

$$[F(x), f(y)]z + [F(x), \tilde{f}(y)]f(z) \in Z(R), \text{ for all } x, y \in R.$$

That is

$$[F(x), \tilde{f}(y)] \in Z(R), \text{ for all } x, y \in R. \tag{41}$$

Taking xz instead of x in (40), with $z \in Z(R)\backslash\{0\}$ and $f(z) \neq 0$, we obtain

$$[F(x), f(y)]z + [\tilde{f}(x), f(y)]f(z) \in Z(R) \tag{42}$$

$$[F(x), f(y)]\tilde{f}(z) + F(x)[\tilde{f}(z), f(y)] + [x, f(y)]f(z) \in Z(R) \tag{43}$$

for all $x, y \in R$.

Relation (42) reduces to $[\tilde{f}(x), f(y)] \in Z(R)$, for all $x, y \in R$, commuting equation (43) with $\tilde{f}(z)$, we obtain

$$\big[[x, f(y)], \tilde{f}(z)\big] = 0 \text{ for all } x, y \in R.$$

Substituting $f(y)x$ for x, we find that

$$[f(y), \tilde{f}(z)]R[x, f(y)] = 0 \quad \text{for all } x, y \in R.$$

Invoking Brauer trick, we get either $[f(y), \tilde{f}(z)] = 0$ or $[x, f(y)] = 0$, for all $x, y \in R$. The second case along with Theorem 2.3 give the commutativity of R. Suppose that

$$[f(y), \tilde{f}(z)] = 0, \text{ for all } x, y \in R.$$

Then expression (43) reduces to

$$[F(x), f(y)]\tilde{f}(z) + [x, f(y)]f(z) \in Z(R), \text{ for all } x, y \in R. \tag{44}$$

In particular for $x = \tilde{f}(z)$

$$[F(\tilde{f}(z)), f(y)]\tilde{f}(z) \in Z(R), \text{ for all } y \in R.$$

Hence $\tilde{f}(z) \in Z(R)$. Then equation (44) reduces to

$$[x, f(y)] \in Z(R), \text{ for all } x, y \in R.$$

Invoking Theorem 2.3, we get R is commutative. □

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